KMS STATES ON FINITE-GRAPH C*-ALGEBRAS

TSUYOSHI KAJIWARA AND YASUO WATATANI

Abstract. We study KMS states on finite-graph C*-algebras with sinks and sources. We compare finite-graph C*-algebras with C*-algebras associated with complex dynamical systems of rational functions. We show that if the inverse temperature β is large, then the set of extreme β-KMS states is parametrized by the set of sinks of the graph. This means that the sinks of a graph correspond to the branched points of a rational funcition from the point of KMS states. Since we consider graphs with sinks and sources, left actions of the associated bimodules are not injective. Then the associated graph C*-algebras are realized as (relative) Cuntz-Pimsner algebras studied by Katsura. We need to generalize Laca-Neshevyev's theorem of the construction of KMS states on Cuntz-Pimsner algebras to the case that left actions of bimodules are not injective.

KEYWORDS: KMS states, graph C*-algebras, C*-correspondences

AMS SUBJECT CLASSIFICATION: 46L08, 46L55

1. Introduction

KMS states on C*-algebras are originated from the equilibrium states in statistical physics. Olsen-Pederson [26] studied KMS states on Cuntz-algebra $\mathcal{O}_n$ ([2]) of n generators with respect to the gauge action, and they proved that a β-KMS state exists if and only if $\beta = \log n$ and the β-KMS state is unique. Evans [5] extended the result to certain quasi-free automorphisms. Enomoto-Fujii-Watatani [4] studied KMS states on Cuntz-Krieger algebras $\mathcal{O}_A$ ([3]) associated with finite graphs with no sinks nor sources. If the 0-1 matrix A is irreducible and not a permutation, then a β-KMS state exists if and only if $\beta = \log r(A)$ and the β-KMS state is unique, where $r(A)$ is the spectral radius of A. Exel-Laca [7] studied KMS states on Exel-Laca algebras and Toeplitz extensions. They introduced KMS states of finite type and infinite type, which are useful in our study. They showed that there occur phase transitions. Exel [6] considered KMS states on his C^*-algebras by endomorphism with a transfer operator. Kumjian and Renault [14] studied KMS states on C^*-algebras associated with expansive maps.

Pimsner [27] introduced a general construction of C*-algebras through Hilbert C*-bimodules or C^*-correspondances. Many C*-algebras are known to be expressed as Cuntz-Pimsner C*-algebras.

The above results on KMS states are extended to KMS states on C*-algebras associated with subshifts in Matsumoto-Watatani-Yoshida [24] and Cuntz-Pimsner algebras associated with bimodules of finite basis in Pinzari-Watatani-Yonetani [28].

Laca-Neshevyev [23] gave a theorem of construction of KMS states for general Cuntz-Pimsner algebras. Using their theorem, we classified KMS states on C*-algebras associated with the complex dynamical systems on the Riemann sphere $\hat{\mathbb{C}}$ given by iteration of rational functions R and C*-algebras associated with self-similar sets in [10] with Izumi. In particular we showed that there exists a phase transition at $\beta = \log \deg R$. If the inverse temperature $\beta > \log \deg R$, then the set of extreme β-KMS states is parametrized by the set of branched points.

On the other hand, Cuntz-Krieger algebras are generalized as graph C*-algebras associated with general graphs having sinks and sources, which are studied for example in Kumujian-Pask-Raeburn [21], Kumujian-Pask-Raeburn-Renault [22] and Fowler-Laca-Raeburn [8]. They consider relations between graphs and the associated graph C*-algebras. See [29] by I. Raeburn to know a total aspect of graph C*-algebras.

In this paper we study KMS states on finite-graph C*-algebras associated with graphs having sinks and sources. We show that if the inverse temperature β is large, then the set of extreme β-KMS states is parametrized by the set of sinks of the graph. We compare finite-graph C*-algebras with C*-algebras associated with complex dynamical systems of rational functions. Our result suggests that the sinks of a graph correspond to the branched points of a rational funcition from the point of KMS states.

Relative Cuntz-Pimsner algebras are a generalization of Cuntz-Pimsner algebras and are defined in [25] and studied in [9]. As in Katsura [19], the graph C^*-algebras associated with graphs having sources and sinks can be constructed as relative Cuntz-Pimsner algebras of bimodules such that left actions are not injective. Hence we shall generalize Laca-Neshevyev's theorem of KMS states to that of relative Cuntz-Pimsner algebras associated with general C*-correspondences with a countable basis. Our proof is more constructive than that of Laca-Neshevyev. We need to investigate the structure of cores of relative Cuntz-Pimsner algebras to study it.

The contents of the present paper is as follows. In section 2, we present the fundamental matters of C*-correspondences, relative Cuntz-Pimsner algebras and the structure of cores of relative Cuntz-Pimsner algebras. In section 3, we present properties of countable basis, the degree of C*-correspondences. We prove a theorem of construction of KMS states on relative Cuntz-Pimsner algebras which generalize the theorem of Laca-Neshevyev. In section 4, we present a classification of KMS states on finite-graph C*-algebras, and show that sinks correspond to KMS states if the inverse temperature is sufficiently large.

2. C*-correspondences and the structure of Cores

In this section, we present fundamental matters of C*-correspondences, the construction of associated C*-algebras, and investigate the structure of the cores of relative Cuntz-Pimsner algebras using some results of Katsura [19], [20].

Definition 2.1. *Let A be a C^*-algebra. A linear space X is called a Hilbert A-module if the following conditions hold:*

1. *There exist an A-valued hermitian, positive definite inner product $(\cdot|\cdot)_A$ and a right action of A which is compatible with the A-inner product.*
2. *X is complete with respect to the norm $\|x\| = \|(x|x)_A\|^{1/2}$.*

If the linear span of A-inner product is dense in A,then X is called full.

Let A be a C^*-algebra and X a Hilbert A-module. We denote by $\mathcal{L}(X)$ the set of linear operators on X which are adjointable with respect to the A-valued inner product. For x and $y \in X$, put $\theta_{x,y}z = x(y|z)_A$ for $z \in X$. We denote by $\mathcal{K}(X)$ the norm closure of the linear span of $\{\, \theta_{x,y} \,| x, y \in X\}$ in $\mathcal{L}(X)$. If there exists a *-homomorphism ϕ from A to $\mathcal{L}(X)$, then we call the pair (X, ϕ) (or simply X) a C*-correspondence over A. We assume neither that X is full, that ϕ is non-degenerate nor that ϕ is isometric. Let $J_X = \phi^{-1}(\mathcal{K}(X)) \cap (\ker \phi)^{\perp}$, and J be a closed two sided ideal of A contained in J_X.

A representation π of a C*-correspondence (X, ϕ) on a Hilbert space $\mathcal{H}$ consists of representations π_A and π_X of A and X i.e. π_A is a *-homomorphism from A to $B(\mathcal{H})$ and π_X is a linear map from X to $B(\mathcal{H})$ satisfying

$$\pi_X(x)^*\pi_X(y) = \pi_A((x|y)_A), \quad \pi_X(x)\pi_A(a) = \pi_X(xa), \qquad \pi_A(a)\pi_X(x) = \pi_X(\phi(a)x),$$

for x, $y \in X$ and $a \in A$. When π_A is injective, π_X is isometric.

For a representation $\pi = (\pi_A, \pi_X)$ of (X, ϕ), there corresponds a representation π_K of $\mathcal{K}(X)$ satisfying $\pi_K(\theta_{x,y}) = \pi_X(x)\pi_X(y)^*$ [12]. The representation $\pi_X^{(n)}$ of $X^{\otimes n}$, $\pi_K^{(n)}$ of $\mathcal{K}(X^{\otimes n})$ are also defined naturally. We use the notation $\mathcal{K}(X^{\otimes 0}) = A$ and $\pi_K^0 = \pi_A$ for convenience.

Definition 2.2. *(Fowler-Muhly-Raeburn* [9]*, Katsura* [20]*) Let J be a closed two sided ideal of A contained in J_X. A representation $\pi = (\pi_A, \pi_X)$ of a C^*-correspondence (X, ϕ) is said to be J-covariant if*

$$\pi_A(a) = \pi_K(\phi(a)) \quad \textit{for any } a \in J. \tag{1}$$

Let $\pi = (\pi_A, \pi_X)$ be the representation of (X, ϕ) which is universal for all J-covariant representations. The relative Cuntz-Pimsner algebra $\mathcal{O}_X(J) = \mathrm{C}^*(\pi)$ is the C^*-algebra generated by $\pi_A(A)$ and $\pi_X(X)$ for the universal representation π. We note that π_A of the universal representation π is known to be injective (Katsura [19] Proposition 4.11).

Lemma 2.3. *(T.Katsura* [19] *Proposition 3.3) Let $\pi = (\pi_A, \pi_X)$ be a representation of (X, ϕ). Assume that π is J-covariant and π_A is injective. Take a in A. If $\pi_A(a)$ is in $\pi_K(\mathcal{K}(X))$, then a is in J_X and $\pi_A(a) = \pi_K(\phi(a))$.*

Lemma 2.4. *Let $\pi = (\pi_A, \pi_X)$ be a representation of (X, ϕ). Assume that π is J-covariant and π_A is injective. Then for $a \in A$, $\pi_A(a)$ is in $\pi_K(\mathcal{K}(X))$ if and only if a is in J.*

Proof. For $a \in A$, assume that $\pi_A(a) \in \pi_K(\mathcal{K}(X))$. By Lemma 2.3, we have $a \in J_X$ and $\pi_A(a) = \pi_K(\phi(a))$.

By Katsura [20] Corollary 11.4, if (π_A, π_X) is a representation of (X, ϕ) satisfying the equation (1), we have

$$\{a \in A \,|\, \phi(a) \in \mathcal{K}(X),\, \pi_A(a) = \pi_K(\phi(a))\} = J.$$

This shows the conclusion. $\square$

We define subalgebras B_n $n \geq 1$ and B_0 by

$$B_n = \pi_K^{(n)}(\mathcal{K}(X^{\otimes n})), \quad B_0 = \pi_A(A).$$

These are C*-subalgebras of $\mathcal{O}_X(J)$. We put

$$\mathcal{F}^{(n)} = B_0 + B_1 + \cdots + B_n.$$

For integers n, i we introduce the notation (n, i) by

$$(n,i) = \begin{cases} n-i & n \geq 1,\, i \geq 1 \\ n-1 & n \geq 1,\, i = 0 \\ 0 & n = 0,\, i = 0. \end{cases}$$

Let $k \in \mathcal{K}(X^{\otimes i})$ $i \geq 1$. For $\xi_1 \in X^{\otimes i}$, $\xi_2 \in X^{\otimes n-i}$, we define $k \otimes id_{(n,i)}$ by

$$(k \otimes id_{(n,i)})(\xi_1 \otimes \xi_2) = k\xi_1 \otimes \xi_2.$$

Then $k \otimes id_{(n,i)}$ is an element of $\mathcal{L}(X^{\otimes n})$. The notation $a \otimes id_{(n,0)}$ means $\phi(a) \otimes id_{n-1}$. When $n = 0$, $a \otimes id_{(0,0)}$ is a left multiplication representation of A on a Hilbert A-module A.

Lemma 2.5. *For each m, B_m is an ideal in $\mathcal{F}^{(m)}$, and $\mathcal{F}^{(m)}$ is a C*-subalgebra.*

Proof. We assume $1 \leq m \leq n$, $k \in \mathcal{K}(X^{\otimes m})$, $k' \in \mathcal{K}(X^{\otimes n})$. Since $k \otimes id_{(n,m)} \in \mathcal{L}(X^{\otimes n})$, we have $(k \otimes id_{(n,m)})k' \in \mathcal{K}(X^{\otimes n})$. By Katsura [19] Lemma 5.4, we have

$$\pi_K^{(m)}(k)\pi_K^{(n)}(k') = \pi_K^{(n)}((k \otimes id_{(n,m)})k').$$

This shows that B_n is an ideal in $\mathcal{F}^{(n)}$. We can check the case $m = 0$ separately. $\square$

We need to investigate $\mathcal{F}^{(n-1)} \cap B_n$ for a proof of the theorem of constructing KMS states.

Note that $(X^{\otimes n}J)_A$ is a right A-submodule of X_A. By considering the embedding of "rank-one" operators, we may have an inclusion $\mathcal{K}(X^{\otimes n}J) \subset \mathcal{K}(X^{\otimes n})$, and we have $\theta_{\xi\cdot j, \eta\cdot j'} \in K(X^{\otimes n}J)$ for j, $j' \in J$.

Lemma 2.6. *An element $k \in \mathcal{K}(X^{\otimes n})$ is in $\mathcal{K}(X^{\otimes n}J)$ if and only if*

$$(\xi|k\eta)_A \in J \qquad \forall \xi, \eta \in X^{\otimes n}.$$

Proof. We refer Fowler-Muhly-Raeburn [9] Lemma 1.6 and Katsura [20] for quotient modules X_J. The notation $[a]_J \in A/J$ for an element a of a C*-algebra A means the quotient image of $a \in A$ by J.

Since $T \in \mathcal{L}(X)$ leaves XJ invariant, we can consider an operator $[T]_J \in X/XJ = X_J$. The map $k \in K(X^{\otimes n}) \to [k]_J$ is an onto map from $K(X^{\otimes n})$ to $K(X^{\otimes n}{}_J)$, and its kernel is k's such that $k \in \mathcal{K}(X^{\otimes n}J)$ ([20] Lemma 1.6). Then $k \in \mathcal{K}(X^{\otimes n})$ is contained in $\mathcal{K}(X^{\otimes n}J)$ if and only if $[k]_J = 0$. Moreover we have

$$\begin{aligned}(\xi|k\eta)_A \in J \quad &\text{if and only if} \quad [(\xi|k\eta)_A]_J = 0\\ &\text{if and only if} \quad ([\xi]_J|[k]_J[\eta]_J)_{A/J} = 0.\end{aligned}$$

If it holds for each ξ, η, then we have $[k]_J = 0$, and this means $k \in \mathcal{K}(X^{\otimes n}J)$. □

We put $B'_n = \pi_K^{(n)}(\mathcal{K}(X^{\otimes n}J))$, $(n \geq 1)$ and $B'_0 = \pi_A(J)$. For the case $J = J_X$, the following Lemma is presented in Katsura [19]. It also holds for the case $J \subset J_X$.

The following lemmas are T.Katsura [19] Lemma 5.10 and T. Katsura [19] Proposition 5.11 for the case $J = J_X$. The proof for general cases is the same as the case $J = J_X$.

Lemma 2.7. *Let $k \in \mathcal{K}(X^{\otimes n+1})$. Then for an approximate unit $\{u_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{K}(X^{\otimes n})$, we have that $k = \lim_{\lambda\in\Lambda}(u_\lambda \otimes id_1)k$.*

Lemma 2.8. *We have that $\mathcal{F}^{(n)} \cap B_{n+1} \subset B_n$.*

Proposition 2.9. *We have that $B_n \cap B_{n+1} = B'_n$ and $\mathcal{F}^{(n)} \cap B_{n+1} = B'_n$.*

Proof. First we show that $B_n \cap B_{n+1} = B'_n$. This is Katsura [19] Proposition 5.9 for $J = J_X$. Let $n = 0$. Then we have

$$\pi_A(A) \cap B_1 = \pi_A(A) \cap \pi_K(\mathcal{K}(X)),$$

By Lemma 2.4, we have $\pi_A(A) \cap B_1 = \pi_A(J)$. Moreover the proposition holds for $n = 0$ because $B'_0 = \pi_A(J)$. We may assume $n \geq 1$. Let a, $b \in J$, ξ, $\eta \in X^{\otimes n}$. Then we have $\pi_A(a) \in B_1 = \pi_K(\mathcal{K}(X))$, and

$$\begin{aligned}\pi_K^{(n)}(\theta_{\xi a,\eta b}) &= \pi_X^{(n)}(\xi a)\pi_X^{(n)}(\eta b)^*\\ &= \pi_X^{(n)}(\xi)\pi_A(a)\pi_A(b)^*\pi_X^{(n)}(\eta)^*.\end{aligned}$$

Since $\pi_A(a)\pi_A(b)^* \in B_1$, the left hand side is contained in B_{n+1}. Then we have

$$B'_n \subset B_n \cap B_{n+1}.$$

Let $x \in B_n \cap B_{n+1}$. There exists $k \in \mathcal{K}(X^{\otimes n})$ such that $\pi_K^{(n)}(k) = x$. For ξ, $\eta \in X^{\otimes n}$, we have

$$\begin{aligned}\pi_A((\xi|k\eta)_A) &= \pi_X^{(n)}(\xi)^*\pi_K^{(n)}(k)\pi_X^{(n)}(\eta)\\ &= \pi_X^{(n)}(\xi)^*x\pi_X^{(n)}(\eta).\end{aligned}$$

By $x \in B_{n+1}$, the last expression is contained in B_1. Since for ξ, $\eta \in X^{\otimes n}$, $\pi_A((\xi|k\eta)_A) \in B_1$, we have $(\xi|k\eta)_A \in J$ for ξ, $\eta \in X^{\otimes n}$ by Lemma 2.4. Then we have $k \in K(X^{\otimes n}J)$, and we have $x = \pi_K^{(n)}(k) \in B_n'$.

Lastly, we shall show that $\mathcal{F}^{(n)} \cap B_{n+1} = B_n'$. By Lemma 2.8, $\mathcal{F}^{(n)} \cap B_{n+1} \subset B_n$. We have

$$\begin{aligned}\mathcal{F}^{(n)} \cap B_{n+1} &= (\mathcal{F}^{(n)} \cap B_{n+1}) \cap B_n = (\mathcal{F}^{(n)} \cap B_n) \cap B_{n+1} \\ &= B_n \cap B_{n+1} = B_n'.\end{aligned}$$

This completes the proof. □

3. KMS states on relative Cuntz-Pimsner algebras

In this section, we generalize a theorem of the construction of KMS states of Cuntz-Pimsner algebras in Laca-Neshevyev [23] to relative Cuntz-Pimsner algebras.

Let A be a σ-unital C*-algebra and X be a countably generated Hilbert A-module.

Definition 3.1. *A sequence $\{u_i\}_{i=1}^{\infty}$ of a Hilbert (right) C^*-module X over A is called a countable basis (or normalized tight frame) of X if*

$$x = \sum_{i=1}^{\infty} u_i(u_i|x)_A \tag{2}$$

for each $x \in X$, where the right hand side converges in norm.

As in Remark after [13] Proposition 1.2 ,

$$\text{For } a, b \in \mathcal{K}(X), x \in X \text{ with } 0 \le a \le b \le I, \quad \|x - bx\|^2 \le \|x\|\|x - ax\|. \tag{3}$$

This inequality implies that the right hand side of (2) converges unconditionally in the following sense: For every $\varepsilon > 0$, there exists a finite subset F_0 of $\mathbb{N}$ such that for every finite subset F of $\mathbb{N}$ with $F_0 \subset F$ we have

$$\|x - \sum_{i \in F} u_i(u_i|x)_A\| < \varepsilon.$$

Since $\sum_{i\in F} \theta_{u_i,u_i} \le I$ for each finite subset $F \in \mathbb{N}$, it is sufficient to prove (2) for each x in some norm dense subset of X. We often write it as $x = \sum_{i\in\mathbb{N}} u_i(u_i|x)_A$ to express unconditinally convergence. More generally, for any countable set Ω, the notation

$$x = \sum_{i \in \Omega} u_i(u_i|x)_A. \tag{4}$$

makes sense as unconditional convergence.

We can show the following Lemma:

Lemma 3.2. *Let A be a C^*-algebra, Y a C^*-correspondence over A and X a Hilbert A-module. Let $\{u_i\}_{i\in\Omega_1}$ be a countable basis of X and $\{v_j\}_{j\in\Omega_2}$ a countable basis of Y. Then $\{u_i \otimes v_j\}_{(i,j)\in\Omega_1\times\Omega_2}$ is a countable basis of the inner tensor product module $X \otimes_A Y$ of X and Y.*

Proof. Let $\varepsilon > 0$. We fix a nonzero $x \otimes y \in X \otimes_A Y$. Let $\delta = \varepsilon^2/\|x \otimes y\|$. be a positive number. We take a finite subset F of Ω_1 such that

$$\|\sum_{i \in F} u_i(u_i|x)_A - x\| < \frac{\delta}{2\|y\|}.$$

Put s be the cardinality of F. For each i ($i = 1, \ldots, s$) we take a finite subset $G_i \subset \Omega_2$ such that if G' is a finite subset containing G_i then it holds that

$$\|\sum_{j \in G'} v_j(v_j|(u_i|x)_A y)_A - (u_i|x)_A y\| < \frac{\delta}{2s\|u_i\|}.$$

Let G be a finite subset containing $\bigcup_{i=1}^{s} G_i$. Then we have

$$\begin{aligned}
&\|x \otimes y - \sum_{(i,j) \in F \times G} u_i \otimes v_j(u_i \otimes v_j|x \otimes y)_A\| \\
=&\|x \otimes y - \sum_{i \in F} \sum_{j \in G} u_i \otimes v_j(v_j|(u_i|x)_A y)\| \\
\leq&\|x \otimes y - \sum_{i \in F} u_i(u_i|x)_A \otimes y\| + \|\sum_{i \in F} u_i \otimes (u_i|x)_A y - \sum_{i \in F} \sum_{j \in G} u_i \otimes v_j(v_j|(u_i|x)_A y)_A\| \\
\leq&\|x - \sum_{i \in F_0} u_i(u_i|x)_A\|\|y\| + \sum_{i \in F} \|u_i\| \|\sum_{j \in G} v_j(v_j|(u_i|x)_A y)_A - (u_i|x)_A y\| \\
<&\delta.
\end{aligned}$$

Using (3), for each finite subset H of $\Omega_1 \times \Omega_2$ such that $H \supset F \times G$ we have that

$$\|x \otimes y - \sum_{(i,j) \in H} u_i \otimes v_j(u_i \otimes v_j|x \otimes y)_A\| < \varepsilon.$$

Hence $x \otimes y = \sum_{(i,j) \in \Omega_1 \times \Omega_2} u_i \otimes v_j(u_i \otimes v_j|x \otimes y)_A$. If $z = \sum_{p \text{ finite}} x_p \otimes y_p$, then

$$\sum_{(i,j) \in \Omega_1 \times \Omega_2} u_i \otimes v_j(u_j \otimes v_j|z)_A = z.$$

Since the subset of elements of the form $\sum_{p \text{ finite}} x_p \otimes y_p$ is dense in $X \otimes_A Y$, $\{u_i \otimes v_j)\}_{\Omega_1 \times \Omega_2}$ constitute a basis of $X \otimes_A Y$. □

We fix a C*-correspondence X over a C*-algebra A, and a countable basis $\{u_i\}_{i=1}^{\infty}$ of X. Let J be a closed two sided ideal of A which is contained in J_X. $\mathcal{O}_X(J)$ denotes the relative Cuntz-Pimsner algebra constructed from X and J in section 2.

Lemma 3.3. *Let τ be a tracial state on A. Then the possibly infinite positive number $\sup_n \sum_{i=1}^{n} \tau((u_i|u_i)_A)$ does not depend on the choice of a countable basis $\{u_i\}_{i=1}^{\infty}$.*

Proof. Let $\{v_i\}_{j=1}^{\infty}$ be another countable basis of X. Since $u_i = \lim_{m\to\infty} \sum_{j=1}^{m} v_j(v_j|u_i)_A$, we have

$$\begin{aligned}
\sup_n \sum_{i=1}^{n} \tau((u_i|u_i)_A) &= \sup_n \lim_{m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \tau((v_j(v_j|u_i)_A|u_i)_A) \\
&= \sup_n \sup_m \sum_{i=1}^{n}\sum_{j=1}^{m} \tau((v_j|u_i)_A^*(v_j|u_i)_A) = \sup_m \sup_n \sum_{i=1}^{n}\sum_{j=1}^{m} \tau((v_j|u_i)_A(v_j|u_i)_A^*) \\
&= \sup_m \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \tau((v_j|u_i)_A(u_i|v_j)_A) = \sup_m \lim_{n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} \tau((v_j|u_i(u_i|v_j)_A)_A) \\
&= \sup_m \sum_{j=1}^{m} \tau((v_j|v_j)_A).
\end{aligned}$$

□

Therefore we may put $d_\tau = \sup_n \sum_{i=1}^{n} \tau((u_i|u_i)_A) \leq \infty$.

We denote by $\mathcal{T}(A)$ the set of tracial states on A.

Definition 3.4. *The degree $d(X)$ of a C^*-correspondence X is defined by $d(X) := \sup\{d_\tau | \tau \in \mathcal{T}(A)\}$. We say that X is of finite-degree type if $d(X) < \infty$.*

Lemma 3.5. *If A is commutative, then $d(X) = \sup_n \| \sum_{i=1}^{n} (u_i|u_i)_A\|$ for any countable basis $\{u_i\}_{i=1}^{\infty}$.*

Proof. We assume that A is commutative. By the method similar as in the proof of Lemma 3.3, we can show that $\sup_n \| \sum_{i=1}^{n}(u_i|u_i)_A\|$ does not depend on the choice of a countable basis $\{u_i\}_{i=1}^{\infty}$.

Since

$$\sum_{i=1}^{n} \tau((u_i|u_i)_A) = \tau(\sum_{i=1}^{n}(u_i|u_i)_A) \leq \| \sum_{i=1}^{n}(u_i|u_i)_A)\|,$$

we have $d(X) \leq \sup_n \| \sum_{i=1}^{n}(u_i|u_i)_A)\|$.

We fix a countable basis $\{u_i\}_{i=1}^{\infty}$. For every $\varepsilon > 0$, there exists an n_0 such that for each $n \geq n_0$,

$$\| \sum_{i=1}^{n}(u_i|u_i)_A\| > \sup_n \| \sum_{i=1}^{n}(u_i|u_i)_A\| - \varepsilon.$$

There exists a tracial state τ such that

$$\tau(\sum_{i=1}^{n}(u_i|u_i)_A) > \| \sum_{i=1}^{n}(u_i|u_i)_A\| - \varepsilon.$$

Thus we have $d(X) \geq \sup_n \| \sum_{i=1}^{n}(u_i|u_i)_A\|$. □

Let R be a rational function of degree N and A a commutative C*-algebra $C(\hat{\mathbb{C}})$. Consider a C*-correspondence X over A associated with the complex dynamical

system given by R on $\hat{\mathbb{C}}$. As described in [11], we can choose a concrete countable basis such that we can compute explicitly as

$$\sum_{i=1}^{n}(u_i|u_i)_A(y) = \#\{R^{-1}(y)\}.$$

This equation is also shown in [16] for any basis. Thus we have

$$\sup_n \|\sum_{i=1}^{\infty}(u_i|u_i)_A\| = N.$$

Therefore the degree of X coincides with the degree of R. Similar formulas hold for the case of self-similar maps.

Let Y be a C*-correspondence over a C*-algebra B of finite-degree type with $d(Y) = N$. Let $\beta > \log N$. For a tracial state τ on B, we can define a bounded linear functional $\hat{\tau}_1$ on $\mathcal{L}(Y)$ by

$$\hat{\tau}_1(k) = e^{-\beta}\sum_{i=1}^{\infty}\tau((u_i|Tu_i)_A),$$

for $T \in \mathcal{L}(Y)$.

We need an elementary fact as follows:

Lemma 3.6. *$\hat{\tau}_1$ is a trace and does not depend on the choice of a basis $\{u_i\}_{i=1}^{\infty}$.*

Proof. Let $\{v_j\}_{j=1}^{\infty}$ be another basis of X, and T be a positive element in $\mathcal{L}(Y)$. As in the proof of Lemma 3.3, we have

$$\begin{aligned}
\sup_n \sum_{i=1}^{n} e^{-\beta}\tau((T^{1/2}u_i|T^{1/2}u_i)_A) &= \sup_n \lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} e^{-\beta}\tau((v_j(v_j|T^{1/2}u_i)_A|T^{1/2}u_i)_A) \\
&= \sup_m \lim_{n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} e^{-\beta}\tau((v_j|T^{1/2}u_i(T^{1/2}u_i|v_j)_A)_A) \\
&= \sup_m \lim_{n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} e^{-\beta}\tau((T^{1/2}v_j|u_i(u_i|T^{1/2}v_j)_A)_A) \\
&= \sup_m \sum_{j=1}^{m} e^{-\beta}\tau((T^{1/2}v_j|T^{1/2}v_j)_A).
\end{aligned}$$

This shows that the definition of $\hat{\tau}_1$ does not depend on the choice of basis. Let U be a unitary in $\mathcal{L}(Y)$. Then $\{Uu_i\}_{i=1}^{\infty}$ is also a basis of Y. For $T \in \mathcal{L}(Y)$, we have

$$\sum_{i=1}^{\infty} e^{-\beta}\tau(Uu_i|TUu_i)_A = \sum_{i=1}^{\infty} e^{-\beta}\tau(u_i|Tu_i)_A.$$

Then $\hat{\tau}_1(U^*TU) = \hat{\tau}_1(T)$, and it follows that $\hat{\tau}_1$ is a trace. $\square$

Let I be an closed two sided ideal of a C*-algebra B, and φ be a state on I. Consider the GNS representation $(\pi_\varphi, H_\varphi, \xi_\varphi)$. Let $\pi : A \to B(H_\varphi)$ be the extension of π_φ to A. Recall that the canonical extension $\overline{\varphi}$ of φ to B is defined as $\overline{\varphi}(a) = (\pi(a)\xi_\varphi, \xi_\varphi)$. Then $\overline{\varphi}(a) = \lim_i \varphi(ae_i)$, for any approximate unit $\{e_i\}_i$ in I as in [1] Prop. 6.4.16.

Let τ_1 be the restriction of $\hat{\tau}_1$ on $\mathcal{K}(Y)$. We note that the C*-algebra $\mathcal{K}(Y)$ is a closed two sided ideal of $\mathcal{L}(Y)$.

Lemma 3.7. *The canonical extension $\overline{\tau}_1$ of τ_1 to $\mathcal{L}(Y)$ is given by $\hat{\tau}_1$.*

Proof. We note that $\hat{\tau}_1(\theta_{x,y}) = e^{-\beta}\tau((y|x)_A)$. If $\{u_i\}_{i=1}^\infty$ is a basis of Y, then $\{\theta_{u_i,u_i}\}_{i=1}^\infty$ is an approximate unit in $\mathcal{K}(Y)$. Therefore for $T \in \mathcal{L}(Y)$, we have

$$\begin{aligned}\overline{\tau}_1(T) &= \lim_{m\to\infty}\sum_j^m \tau_1(T\theta_{u_j,u_j}) = \lim_{m\to\infty}\sum_j^m \tau_1(\theta_{Tu_j,u_j}) \\ &= \lim_{m\to\infty}\sum_{j=1}^m e^{-\beta}\tau((u_j|Tu_j)_A) = \hat{\tau}_1(T)\end{aligned}$$

□

Let A be a C*-algebra and X be a C*-correspondence over A with a countable basis $\{u_i\}_{i=1}^\infty$. Since we use tensor products of correspondences and their bases frequently, we use the notations of multi index. Namely, for $\mathbf{p} = (i_1, i_2, \dots, i_n) \in \mathbb{N}^n$, we write $\mathbf{u_p} = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n}$.

We assume that X is of finite-degree type. We can also define a bounded tracial linear functional $\hat{\tau}^{(n)}$ on $\mathcal{L}(X^{\otimes n})$ and its restriction $\tau^{(n)}$ to $\mathcal{K}(X^{\otimes n})$ using the Hilbert A-module $X^{\otimes n}$ and its basis $\{\mathbf{u_p}\}_{\mathbf{p}\in\mathbb{N}^n}$ as

$$\hat{\tau}^{(n)}(T) = e^{-n\beta}\sum_{\mathbf{p}\in\mathbb{N}^n}\tau((\mathbf{u_p}|T\mathbf{u_p})_A) \qquad \text{for } T \in \mathcal{L}(X^{\otimes n}).$$

Definition 3.8. *Let J be a closed two-sided ideal of A such that $J \subset J_X$, and β be a positive real number. A tracial state τ on A satisfies β-condition if it satisfies the following two conditions:*

($\beta 1$) $\sum_{i=1}^\infty \tau((u_i|\phi(a)u_i)_A) = e^\beta\tau(a) \quad \forall a \in J$,

($\beta 2$) $\sum_{i=1}^\infty \tau((u_i|\phi(a)u_i)_A) \le e^\beta\tau(a) \quad \forall a \in A^+$.

Since B_n is isomorphic to $\mathcal{K}(X^{\otimes n})$ by $\pi_K^{(n)}$ for each n, we can define a bounded linear tracial functional $\sigma^{(n)}$ on B_n by

$$\sigma^{(n)} = \tau^{(n)} \circ (\pi_K^{(n)})^{-1}.$$

For convenience, we put $\tau^{(0)} = \tau$, $\sigma^{(0)} = \tau \circ {\pi_A}^{-1}$.

Proposition 3.9. *We assume that a tracial state τ on A satisfies ($\beta 1$). Then, for $x \in \mathcal{F}^{(n)} \cap B_{n+1} = B_n \cap B_{n+1}$, we have*

$$\sigma^{(n+1)}(x) = \sigma^{(n)}(x).$$

Proof. We put $\mathbf{p} = (i_1, i_2, \ldots, i_n)$, $\mathbf{u_p} = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n}$, $\mathbf{p'} = (i_1, i_2, \ldots, i_n, i_{n+1})$ and $\mathbf{u_{p'}} = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n} \otimes u_{i_{n+1}} = \mathbf{u_p} \otimes u_{i_{n+1}}$.

Due to $x \in \mathcal{F}^{(n)} \cap B_{n+1} = B_n \cap B_{n+1} = B_n'$, we can write as $x = \pi_K^{(n)}(k)$, $k \in K(X^{\otimes n}J)$, which shows $(\mathbf{u_p}|k\mathbf{u_p})_A \in J$ for each $\mathbf{u_p}$.

On the other hand, we can write $x = \pi_K^{(n+1)}(k')$, $k' \in K(X^{\otimes n+1})$ because $x \in B_{n+1}$. Then we have

$$\begin{aligned}\pi_A((\mathbf{u_{p'}}|k'\mathbf{u_{p'}})_A) =&\pi_X(u_{i_{n+1}})^* \cdots \pi_X(u_{i_1})^* x \pi_X(u_{i_1}) \cdots \pi_X(u_{i_{n+1}}) \\ =&\pi_X(u_{i_{n+1}})^* \left(\pi_X(u_{i_n})^* \cdots \pi_X(u_{i_n})^* x \pi_X(u_{i_1}) \cdots \pi_X(u_{i_n})\right) \pi_X(u_{i_{n+1}}) \\ =&\pi_X(u_{i_{n+1}})^* \left(\pi_A(\mathbf{u_p}|k\mathbf{u_p})_A)\right) \pi_X(u_{i_{n+1}}) \\ =&\pi_X(u_{i_{n+1}})^* \pi_X(\phi((\mathbf{u_p}|k\mathbf{u_p})_A) u_{i_{n+1}}) \\ =&\pi_A((u_{i_{n+1}}|\phi((\mathbf{u_p}|k\mathbf{u_p})_A) u_{i_{n+1}})_A).\end{aligned}$$

Then we have

$$\tau^{(n+1)}(k') = e^{-(n+1)\beta} \sum_{\mathbf{p} \in \mathbb{N}^n} \sum_{i_{n+1}=1}^{\infty} \tau((u_{i_{n+1}}|\phi((\mathbf{u_p}|k\mathbf{u_p})_A) u_{i_{n+1}})_A).$$

Using $(\beta 1)$

$$\begin{aligned}\tau^{(n+1)}(k') &= e^{-n\beta} \sum_{\mathbf{p} \in \mathbb{N}^n} \tau((\mathbf{u_p}|k\mathbf{u_p})_A) \\ &= \tau^{(n)}(k).\end{aligned}$$

By this, we have $\sigma^{(n+1)}(x) = \sigma^{(n)}(x)$ for $x \in \mathcal{F}^{(n)} \cap B_{n+1}$. $\square$

We assume that τ satisfies $(\beta 2)$. $\tau^{(n+1)}$ is a tracial bounded linear functional on $\mathcal{K}(X^{\otimes n+1})$.

We assume $n \geq 1$. We denote by $\mathcal{F}(\Sigma)$ the set of finite subsets of Σ. Let $e_F = \sum_{\mathbf{p} \in F} \theta_{\mathbf{u_p}, \mathbf{u_p}}$ for a finite subset F of $\mathbb{N}^n$. Then $\{e_F\}_{F \in \mathcal{F}(\mathbb{N}^n)}$ is an approximate unit of $\mathcal{K}(X^{\otimes n})$. The canonical extension $\overline{\tau^{(n)}}$ of $\tau^{(n)}$ to $\mathcal{L}(X^{\otimes n})$ satisfies

$$\overline{\tau^{(n)}}(T) = e^{-n\beta} \lim_F \sum_{\mathbf{p} \in F} \tau((\mathbf{u_p}|Te_F\mathbf{u_p})_A),$$

where $T \in \mathcal{L}(X^{\otimes n})$, and it is expressed by Lemma 3.7 as

$$\overline{\tau^{(n)}}(T) = e^{-n\beta} \sum_{\mathbf{q} \in \mathbb{N}^n} \tau((\mathbf{u_q}|T\mathbf{u_q})_A)$$

for $T \in \mathcal{L}(X^{\otimes n})$. Then the following Lemma holds.

Lemma 3.10. *We assume that τ satisfis $(\beta 2)$. Let $n \geq 1$ and $0 \leq i \leq n$. For $k \in \mathcal{K}(X^{\otimes i})$. we have*

$$\overline{\tau^{(n)}}(k \otimes id_{(n,i)}) = e^{-n\beta} \sum_{\mathbf{q} \in \mathbb{N}^n} \tau((\mathbf{u_q}|(k \otimes id_{(n,i)})\mathbf{u_q})_A).$$

Since B_{n+1} is an ideal of $\mathcal{F}^{(n+1)}$, there exists the canonical extension $\overline{\sigma^{(n+1)}}$ on $\mathcal{F}^{(n+1)}$. For a finite subset F of $\mathbb{N}^{n+1}$, we put $\hat{e}_F = \sum_{\mathbf{q}\in F} \pi_X^{(n+1)}(\mathbf{u_q})\pi_X^{(n+1)}(\mathbf{u_q})^* = \sum_{\mathbf{q}\in F} \pi_K^{(n+1)}(\theta_{\mathbf{u_q},\mathbf{u_q}})$. Then $\{\hat{e}_F\}_{F\in\mathcal{F}(\mathbb{N}^{n+1})}$ is an approximate unit of B_{n+1}. Then we have

$$\overline{\sigma^{(n+1)}}(x) = \lim_F \sigma^{(n+1)}(x\hat{e}_F),$$

for $x \in \mathcal{F}^{(n+1)}$.

Let $x \in B_i$ for $0 \le i \le n$. We write as $x = \pi_K^{(i)}(k)$ where $k \in \mathcal{K}(X^{\otimes i})$. Then we have

$$x\hat{e}_F = \sum_{\mathbf{q}\in F} \pi_K^{(n+1)}((k \otimes id_{(n+1,i)})\theta_{\mathbf{u_q},\mathbf{u_q}}).$$

Using this,

$$\begin{aligned}\overline{\sigma^{(n+1)}}(x) =& \lim_F \sigma^{(n+1)}(x\hat{e}_F)\\ =& \lim_F \tau^{(n+1)}\left((k \otimes id_{(n+1,i)}) \sum_{\mathbf{q}\in F} \theta_{\mathbf{u_q},\mathbf{u_q}}\right) = \overline{\tau^{(n+1)}}(k \otimes id_{(n,i)}).\end{aligned}$$

Lemma 3.11. *We assume that τ satisfis ($\beta 2$). Let $x \in \mathcal{F}^{(n)}$ with $x = \sum_{i=0}^n x_i$, where $x_i \in B_i$. Take $k_i \in \mathcal{K}^{(i)}(X^{\otimes i})$ such that $x_i = \pi_K^{(i)}(k_i)$. Then we have*

$$\overline{\sigma^{(n+1)}}(x) = e^{-(n+1)\beta} \sum_{\mathbf{q}\in\mathbb{N}^{n+1}} \tau((\mathbf{u_q}|\sum_{i=1}^n (k_i \otimes id_{(n+1,i)})\mathbf{u_q})_A).$$

Proof. Using Lemma 3.10, we have

$$\begin{aligned}\overline{\sigma^{(n+1)}}(x) =& \sum_{i=0}^n \overline{\sigma^{(n+1)}}(x_i) = \sum_{i=0}^n \overline{\tau^{(n+1)}}(k_i \otimes id_{(n+1,i)})\\ =& e^{-(n+1)\beta} \sum_{\mathbf{q}\in\mathbb{N}^{n+1}} \tau((\mathbf{u_q}|\sum_{i=1}^n (k_i \otimes id_{(n+1,i)})\mathbf{u_q})_A).\end{aligned}$$

□

Proposition 3.12. *We assume that τ satisfies ($\beta 2$). For $x \in (F^{(n)})^+$, we have*

$$\overline{\sigma^{(n+1)}}(x) \le \overline{\sigma^{(n)}}(x).$$

Proof. We take $x \in (F^{(n)})^+$. Then we can write as $x = y^*y$ where $y \in \mathcal{F}^{(n)}$. We also write as $y = \sum_{i=0}^n y_i$ where $y_i \in B_i$, and write as $y_i = \pi_K^{(i)}(h_i)$, $h_i \in \mathcal{K}(X^{\otimes i})$.

Then, by Lemma 5.4 in [19], we have

$$
\begin{aligned}
x &= \sum_{i=0}^{n} \sum_{j=0}^{n} y_i^* y_j = \sum_{i=0}^{n} \sum_{j=0}^{n} \pi_K^{(i)}(h_i)^* \pi_K^{(j)}(h_j) \\
&= \sum_{i=0}^{n} \pi_K^{(i)} \left(\sum_{j=0}^{i} (h_j \otimes id_{(i,j)})^* h_i + h_i^* \sum_{j=0}^{i-1} (h_j \otimes id_{(i,j)}) \right) = \sum_{i=0}^{n} \pi_K^{(i)}(k_i),
\end{aligned}
$$

where

$$
k_i = \sum_{j=0}^{i} (h_j \otimes id_{(i,j)})^* h_i + h_i^* \sum_{j=0}^{i-1} (h_j \otimes id_{(i,j)}).
$$

We put

$$
k = \sum_{i=0}^{n} k_i \otimes id_{(n,i)}.
$$

$$
\begin{aligned}
k &= \sum_{i=0}^{n} \left(\sum_{j=0}^{i} (h_j \otimes id_{(i,j)})^* h_i + h_i^* \sum_{j'=0}^{i-1} h_{j'} \otimes id_{(i,j')} \right) \otimes id_{(n,i)} \\
&= \sum_{i=0}^{n} \left(\sum_{j=0}^{i} (h_j \otimes id_{(n,j)})^* (h_i \otimes id_{(n,i)}) + \sum_{j'=0}^{i-1} (h_i \otimes id_{(n,i)})^* (h_{j'} \otimes id_{(n,j')}) \right) \\
&= \left(\sum_{i=0}^{n} (h_i \otimes id_{(n,i)}) \right)^* \left(\sum_{j=0}^{n} (h_j \otimes id_{(n,j)}) \right) \\
&\geq 0.
\end{aligned}
$$

As in the proof of Proposition 3.9, we put $\mathbf{p} = (i_1, i_2, \dots, i_n)$, $\mathbf{u_p} = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n}$, $\mathbf{p}' = (i_1, i_2, \dots, i_n, i_{n+1})$ and $\mathbf{u_{p'}} = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n} \otimes u_{i_{n+1}} = \mathbf{u_p} \otimes u_{i_{n+1}}$.

We prepare the following: For $k \in \mathcal{K}(X^{\otimes i})$ $(0 \leq i \leq n)$, we have

$$
\begin{aligned}
& e^{-(n+1)\beta} \sum_{\mathbf{p}' \in \mathbb{N}^{n+1}} \tau((\mathbf{u_{p'}} | (k \otimes id_{(n+1,i)}) \mathbf{u_{p'}})_A) \\
=& e^{-(n+1)\beta} \sum_{\mathbf{p} \in \mathbb{N}^n} \sum_{i_{n+1}=1}^{\infty} \tau((\mathbf{u_p} \otimes u_{i_{n+1}} | (k \otimes id_{(n+1,i)}) (\mathbf{u_p} \otimes u_{i_{n+1}}))_A) \\
=& e^{-(n+1)\beta} \sum_{\mathbf{p} \in \mathbb{N}^n} \sum_{i_{n+1}=1}^{\infty} \tau((u_{i_{n+1}} | \phi((\mathbf{u_p} | (k \otimes id_{(n,i)}) \mathbf{u_p})_A) u_{i_{n+1}})_A).
\end{aligned}
$$

If τ satisfies $(\beta 2)$ and $T \in \mathcal{L}(X^{\otimes n})$ is positive, we have

$$
\sum_{i_{n+1}=1}^{\infty} \tau((u_{i_{n+1}} | (\mathbf{u_p} | T \mathbf{u_p})_A u_{i_{n+1}})_A) \leq e^{\beta} \tau((\mathbf{u_p} | T \mathbf{u_p})_A).
$$

Using these, we prove the Proposition. Let $x \in (\mathcal{F}^{(n)})^+$. We express $x = \sum_{i=1}^n x_i$ where $x_i \in B_i$. For x_i we take k_i such that $x_i = \pi_K^i(k_i)$ and write as $k = \sum_{i=0}^n (k_i \otimes id_{(n+1,i)})$. Then by Lemma 3.7 and by the fact that τ satisfies $(\beta 2)$, we have

$$\begin{aligned}
\overline{\sigma^{(n+1)}}(x) =& e^{-(n+1)\beta} \sum_{\mathbf{p}' \in \mathbb{N}^{n+1}} \tau((\mathbf{u}_{\mathbf{p}'} | \sum_{i=0}^n (k_i \otimes id_{(n+1,i)}) \mathbf{u}_{\mathbf{p}'})_A) \\
=& e^{-(n+1)\beta} \sum_{\mathbf{p}' \in \mathbf{n}^{n+1}} \tau((\mathbf{u}_{\mathbf{p}'} | (k \otimes id_1) \mathbf{u}_{\mathbf{p}'})_A) \\
=& e^{-(n+1)\beta} \sum_{\mathbf{p} \in \mathbb{N}^n} \sum_{i_{n+1}=1}^{\infty} \tau((\mathbf{u}_{\mathbf{p}} \otimes u_{i_{n+1}} | ((k\mathbf{u}_{\mathbf{p}}) \otimes u_{i_{n+1}}))_A) \\
=& e^{-(n+1)\beta} \sum_{\mathbf{p} \in \mathbb{N}^n} \sum_{i_{n+1}=1}^{\infty} \tau((u_{i_{n+1}} | \phi((\mathbf{u}_{\mathbf{p}} | k\mathbf{u}_{\mathbf{p}})_A) u_{i_{n+1}})_A) \\
\leq& e^{-n\beta} \sum_{\mathbf{p} \in \mathbb{N}^n} \tau((\mathbf{u}_{\mathbf{p}} | k\mathbf{u}_{\mathbf{p}})_A) \\
=& \overline{\sigma^{(n)}}(x).
\end{aligned}$$

□

Let A be a C*-algebra, α be an automorphic action of one dimensional torus $\mathbb{T}$ on A. A^{anal} denotes the set $a \in A$ such that $t \to \alpha_t(a)$ has an analytic extension to $\mathbb{C}$.

Definition 3.13. *Let $\beta > 0$. A state φ of A is called a β-KMS state on A with respect to α if*

$$\varphi(x\alpha_{it}(y)) = \varphi(yx)$$

*for $x \in A$ and $y \in D$, where D is a dense *-subalgebra contained in A^{anal}.*

We denote by E the conditional expectation of $\mathcal{O}_X(J)$ onto the fixed point algebra $\mathcal{O}_X(J)^{\mathbb{T}}$ by the gauge action. We denote by $\mathcal{O}_X(J)^{(n)}$ the n-spectral subspace with respect to the gauge action.

Lemma 3.14. *([28] Proposition 1.3) Fix $\beta > 0$. If φ is a β-KMS state on $\mathcal{O}_X(J)$, then for x, $y \in \mathcal{O}_X(J)^{(n)}$,*

$$\varphi(x^*y) = e^{n\beta}\varphi(yx^*). \tag{5}$$

Conversely, if a tracial state φ on $\mathcal{O}_X(J)^{\mathbb{T}}$ satisfies the equation (5) for x, $y \in \mathcal{O}_X(J)^{(n)}$,then $\tau \circ E$ is a β-KMS state on $\mathcal{O}_X(J)$. The correspondence is one to one and conserves extreme points.

Lemma 3.15. *(Exel-Laca [7] Proposition 12.5) Let B be a unital C*-algebras, A a C*-subalgebra of B containing unit, and I a closed two sided ideal of B such that $B = A + I$. Let φ be a state on A and ψ a positive linear functional on I. We assume that $\varphi(x) = \psi(x)$ for $x \in A \cap I$ and that $\overline{\psi(x)} \leq \varphi(x)$ for $x \in A$. Then there exists a unique state Φ on B such that $\Phi|_A = \varphi$ and $\Phi|_I = \psi$.*

Corollary 3.16. *Let B be a unital C^*-algebras, A be a C^*-subalgebra of B containing unit, and I be a closed two sided ideal of B such that $B = A + I$. Let φ be a tracial state on A and ψ a trace on I. We assume that $\varphi(x) = \psi(x)$ for $x \in A \cap I$ and that $\overline{\psi(x)} \leq \varphi(x)$ for $x \in A$. Then there exists a unique tracial state Φ on B such that $\Phi|_A = \varphi$ and $\Phi|_I = \psi$.*

Proof. Let Φ be the state extension on B constructed in Lemma 3.15. All we have to show is that Φ is tracial. Consider GNS representation $(\pi_\psi, H_\psi, \xi_\psi)$ of I. Let $\pi : B \to B(H_\psi)$ be the extension of π_ψ. The canonical extension $\overline{\psi}$ of ψ to B is defined as $\overline{\psi}(b) = (\pi(b)\xi_\psi \mid \xi_\psi)$ for $b \in B$. Define a state ψ' on the von Neumann algebra $\pi(I)''$ by $\psi'(m) = (m\xi_\psi \mid \xi_\psi)$, for $m \in \pi(I)''$. Since ψ is tracial, ψ' is also tracial. Since $\overline{\psi}(b) = (\pi(b)\xi_\psi \mid \xi_\psi)$, the canonical extension $\overline{\psi}$ is also tracial. Hence for $a, b \in A$ and $x, y \in I$, we have

$$\begin{aligned}\Phi((a+x)(b+y)) =& \Phi(ab + xb + ay + xy) = \varphi(ab) + \overline{\psi}(xb + ay + xy) \\ =& \varphi(ba) + \overline{\psi}(bx + ya + yx) = \Phi((b+y)(a+x)),\end{aligned}$$

because φ and $\overline{\psi}$ are tracial. Thus Φ is also tracial.

□

Under these preparations, we shall generalize Laca-Neshevyev's theorem of the construction of KMS states on Cuntz-Pimsner algebras as follows:

Theorem 3.17. *Let X be a C^*-correspondence over A of finite-degree type with degree $N = d(X)$ and $\{u_i\}_{i=1}^\infty$ a countable basis of X. Let J be an ideal of A contained in J_X. Let $\beta > 0$. Let φ be a β-KMS state on a relative Cuntz-Pimsner algebra $\mathcal{O}_X(J)$ with respect to the gauge action γ. Then the restriction of φ to $\pi_A(A)$ is a tracial state on A satisfying β-condition. Conversely,a tracial state on A satisfying β-condition extends to a β-KMS state on $\mathcal{O}_X(J)$. The correspondence between the β-KMS states on $\mathcal{O}_X(J)$ and the tracial states on A satisfying β-condition given by $\varphi \to \varphi|_{\pi_A(A)} \circ \pi_A$ is bijective and affine.*

Proof. Let φ be a β-KMS state on $\mathcal{O}_X(J)$. The restriction of φ to $\mathcal{O}_X(J)^{\mathbb{T}}$ is a tracial state and satisfies the condition (5) in Lemma 3.14. For $a \in J$, we have $\phi(a) \in \mathcal{K}(X)$ and $\pi_A(a) = \pi_K(\phi(a))$. Since the equation

$$\sum_{i=1}^n \pi_X(u_i)\pi_X(u_i)^*\pi_A(a) = \pi_K(\sum_{i=1}^n \theta_{u_i,u_i}\phi(a))$$

holds and $\{\sum_{i=1}^n \theta_{u_i,u_i}\}_{n=1}^\infty$ is an approximate unit of $\mathcal{K}(X)$, we have

$$\lim_{n\to\infty} \sum_{i=1}^n \pi_X(u_i)\pi_X(u_i)^*\pi_A(a) = \pi_A(a).$$

Using this, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\varphi(\pi_A((u_i|\phi(a)u_i)_A)) =&\sum_{i=1}^{\infty}\varphi(\pi_X(u_i)^*\pi_A(a)\pi_X(u_i))\\ =&e^{\beta}\sum_{i=1}^{\infty}\varphi(\pi_X(u_i)\pi_X(u_i)^*\pi_A(a))\\ =&e^{\beta}\varphi\left(\lim_{n\to\infty}\left(\left(\sum_{i=1}^{n}\pi_X(u_i)\pi_X(u_i)^*\right)\pi_A(a)\right)\right)\\ =&e^{\beta}\varphi(\pi_A(a)).\end{aligned}$$

This shows that $\varphi\circ\pi_A$ satisfies (β1). Let $a\in A^+$. We have

$$\begin{aligned}\sum_{i=1}^{n}(\phi\circ\pi_A)((u_i|\phi(a)u_i)_A) =&\sum_{i=1}^{n}\varphi(\pi_X(u_i)^*\pi_A(a)\pi_X(u_i))\\ =&e^{\beta}\sum_{i=1}^{n}\varphi(\pi_A(a)\pi_X(u_i)\pi_X(u_i)^*)\\ =& e^{\beta}\varphi\left(\pi_A(a)^{1/2}\left(\sum_{i=1}^{n}\pi_X(u_i)\pi_X(u_i)^*\right)\pi_A(a)^{1/2}\right)\\ \le& e^{\beta}\varphi(\pi_A(a)).\end{aligned}$$

As $n\to\infty$, we can show that $\varphi\circ\pi_A$ satisfies (β2).

Conversely, we take a tracial state τ on A satisfying (β1) and (β2). We construct a tracial state ω on $\mathcal{O}_X(J)^{\mathbb{T}}$ satisfying the condition (5) in Lemma 3.14 and $\omega|_{\pi_A(A)}\circ\pi_A=\tau$ holds.

We construct a tracial state $\omega^{(n)}$ on $\mathcal{F}^{(n)}$ for each natural integer n inductively. We put $\omega^{(0)}=\tau\circ\pi_A^{-1}$. We assume that there exists a tracial state $\omega^{(n)}$ on $\mathcal{F}^{(n)}$ such that

$$\omega^{(n)}|_{B_n}=\tau^{(n)}\circ(\pi_K^{(n)})^{-1}=\sigma^{(n)}$$

and

$$\overline{\sigma^{(n)}}\le\omega^{(n)}\qquad\text{on }\mathcal{F}^{(n-1)}.$$

By Proposition 3.9, for $x\in\mathcal{F}^{(n)}\cap B_{n+1}=B_n\cap B_{n+1}$ we have

$$\sigma^{(n+1)}(x)=\sigma^{(n)}(x)=\omega^{(n)}(x).$$

For $x\in\mathcal{F}^{(n)}$, by Proposition 3.12, we have

$$\overline{\sigma^{(n+1)}}(x^*x)\le\overline{\sigma^{(n)}}(x^*x).\tag{6}$$

By the assumption of induction, for $x\in\mathcal{F}^{(n-1)}$,

$$\overline{\sigma^{(n)}}(x^*x)\le\omega^{(n)}(x^*x).$$

Let $x \in \mathcal{F}^{(n)}$ be written as $x = y + z$ where $y \in \mathcal{F}^{(n-1)}$ and $z \in B_n$. Then we have

$$\begin{aligned}
\overline{\sigma^{(n)}}(x^*x) &= \overline{\sigma^{(n)}}((y+z)^*(y+z)) \\
&= \overline{\sigma^{(n)}}(y^*y + y^*z + z^*y + z^*z) \\
&= \overline{\sigma^{(n)}}(y^*y) + \sigma^{(n)}(y^*z + z^*y + z^*z) \\
&= \overline{\sigma^{(n)}}(y^*y) + \omega^{(n)}(y^*z + z^*y + z^*z) \\
&\leq \omega^{(n)}(y^*y) + \omega^{(n)}(y^*z + z^*y + z^*z) \\
&= \omega^{(n)}(y^*y + y^*z + z^*y + z^*z) \\
&= \omega^{(n)}(x^*x). \qquad (7)
\end{aligned}$$

Using (6) and (7), for $x \in \mathcal{F}^{(n)}$ we have

$$\overline{\sigma^{(n+1)}}(x^*x) \leq \omega^{(n)}(x^*x).$$

By Lemma 3.15, there exists a state $\omega^{(n+1)}$ on $\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + B_{n+1}$ such that $\omega^{(n+1)}|_{\mathcal{F}^{(n)}} = \omega^{(n)}$ and $\omega^{(n+1)}|_{B_{n+1}} = \sigma^{(n+1)}$. Since $\omega^{(n)}$ and $\sigma^{(n+1)}$ are traces, we have that $\omega^{(n+1)}$ is a trace by Lemma 3.16.

We note that that $\overline{\sigma^{(n+1)}} \leq \omega^{(n+1)}$ on $\mathcal{F}^{(n)}$ because $\omega^{(n+1)} = \omega^{(n)}$ on $\mathcal{F}^{(n)}$.

By a mathematical induction argument, there exists a desired sequence $\{\omega^{(n)}\}_{n=1,2,\ldots}$ of tracial states on $\mathcal{F}^{(n)}$. We define ω on $\bigcup_{n=0}^{\infty} \mathcal{F}^{(n)}$ by $\omega|_{\mathcal{F}^{(n)}} = \omega^{(n)}$, and extend it to the closure $\mathcal{O}_X(J)^{\mathbb{T}}$ by continuity. Then ω is a tracial state on $\mathcal{O}_X(J)^{\mathbb{T}}$.

Since $\omega(\pi_A(a) + \pi_K^{(1)}(k_1) + \cdots + \pi_K^{(n)}(k_n)) = \tau(a) + \tau^{(1)}(k_1) + \cdots + \tau^{(n)}(k_n)$ for $a \in A$ and $k_i \in \mathcal{K}(X^{\otimes i})$, ω does not depend on the choice of the basis $\{u_k\}_{k=1}^{\infty}$ we have used in the construction.

From $\omega(\theta_{x_1,\cdots,x_n,y_1,\cdots,y_n}) = e^{-n\beta}\tau((y_1 \otimes \cdots \otimes y_n | x_1 \otimes \cdots \otimes x_n)_A)$, ω satisfies the condition (5) of Lemma 3.14. Let $E : \mathcal{O}_X(J) \to \mathcal{O}_X(J)^{\mathbb{T}}$ be the canonical conditional expectation. Put $\varphi = \omega \circ E$. Then φ is a β-KMS state of $\mathcal{O}_X(J)$ such that its restriction to A is τ. □

4. KMS states on the C*-algebra associated with finite graphs

Cuntz-Krieger algebras are generalized as graph C*-algebras associated with general graphs having sinks and sources, which are studied for example in Kumujian-Pask-Raeburn [21], Kumujian-Pask-Raeburn-Renault [22] and Fowler-Laca-Raeburn [8]. As in Katsura [19] C*-algebras associated with graphs with possibly sources and sinks are expressed as C*-algebras associated with C*-correspondences canonically constructed from graphs. But the left actions are not necessarily injective,

Using the construction and Theorem 3.17 in the preceding section, we can describe KMS states on finite-graph C*-algebras.

Let $E = (E^0, E^1)$ be a finite graph without multiple edges. We denote by s the source map and by r the range map of E. A vertex $v \in E^0$ is called a sink if $s^{-1}(v) = \emptyset$ and $v \in E^0$ is called a source if $r^{-1}(v) = \emptyset$

Definition 4.1. *The graph C^*-algebra $C^*(E)$ of a finite graph E is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v\}_{v\in E^0}$ and partial isometries $\{q_e\}_{e\in E^1}$ with orthogonal ranges, such that $q_e^*q_e = p_{r(e)}$, $q_eq_e^* \le p_{s(e)}$ for $e \in E^1$ and*

$$p_v = \sum_{e\in s^{-1}(v)} q_eq_e^* \qquad \textit{for} \;\; 0 < |s^{-1}(v)|.$$

We put

$$E_r^0 = \{\, v \in E^{(0)} \mid |s^{-1}(v)| > 0 \,\}, \quad E_s^0 = \{\, v \in E^{(0)} \mid |s^{-1}(v)| = 0 \,\}.$$

We note that E_s^0 is the set of sinks of E

Let $A = \mathrm{C}(E^0)$ and $X = \mathrm{C}(E^1)$. For ξ, $\eta \in X$ and $f \in A$, we put

$$(\xi|\eta)_A(v) = \sum_{e\in r^{-1}(v)} \overline{\xi(e)}\eta(e) \qquad \forall v \in E^0$$

$$(\xi f)(e) = \xi(e)f(r(e)) \qquad \forall e \in E^1.$$

Then X is a Hilbert C*-module over A.

We define $\phi(f)$ for $f \in A$ by

$$\phi(f)\xi(e) = f(s(e))\xi(e),$$

where $\xi \in X$. Then ϕ is a *-representation of A in $\mathcal{L}(X_A)$ and (X, ϕ) is a C*-correspondence over A. As in atsura [19], it holds that $\phi^{-1}(K(X)) = \mathrm{C}(E^0) = A$, $\ker(\phi) = \mathrm{C}(E_s^0)$ and $J_X = \mathrm{C}(E_r^0)$. The left action ϕ on X is injective if E has no sinks. The C*-correspondence X is full if E has no sources. We denote by $\mathcal{O}_E$ the (relative) Cuntz-Pimsner algebra of the C*-correspondence X with $J = J_X$. Then $\mathcal{O}_E$ is isomorphic to the graph C*-algebra C*(E) ([18]).

We denote by γ the gauge action of $\mathbb{T}$ on $\mathcal{O}_E$. Let β be a positive number. We consider β-KMS states on the C*-algebra $\mathcal{O}_E$ with respect to the gauge action γ.

We number vertices of E from 1 to n. Let e be an edge such that $s(e) = i$ and $r(e) = j$. Since multiples edges are not permitted, we write e as (i, j). We denote by $\chi_{(i,j)}$ the characteristic function of the edge (i, j). Then $\{\chi_{(i,j)}|(i, j) \in E^1\}$ constitutes a basis of X on A.

By Theorem 3.17, there exists a bijective correspondence between the β-KMS states on $\mathcal{O}_E$ and the tracial states τ on the commutative C*-algebra A such that

$$\sum_{(i,j)\in E^1} \tau((\chi_{(i,j)}|\phi(f)\chi_{(i,j)})_A) = e^\beta\tau(f) \qquad f \in J_X = \mathrm{C}(E_r^0)$$

$$\sum_{(i,j)\in E^1} \tau((\chi_{(i,j)}|\phi(f)\chi_{(i,j)})_A) \le e^\beta\tau(f) \qquad f \in A^+ = \mathrm{C}(E^0)^+.$$

The correspondence is affine.

We denote by $D = [a_{i,j}]_{1\le i,j\le n}$ the adjacency matrix of the graph E i.e.

$$a_{ij} = \begin{cases} 1 & \text{there exists } e \in E^1 \text{ such that } s(e) = j,\, r(e) = i \\ 0 & \text{otherwise,} \end{cases}$$

and we denote by χ_j the characteristic function of the vertex j.

Then we have

$$\begin{aligned}(\chi_{(i,j)}|\phi(\chi_l)\chi_{(i,j)})_A(k) &= \sum_{r(e)=k} \overline{\chi_{(i,j)}(e)}\phi(\chi_l)\chi_{(i,j)}(e) \\ &= \delta_{l,i}\delta_{j,k}a_{k,l}.\end{aligned}$$

We put $f = \sum_{l=1}^{n} f_l\chi_l$. Then we have

$$(\chi_{(i,j)}|\phi(f)\chi_{(i,j)})_A(k) = \sum_{l=1}^{n} f_l\delta_{l,i}\delta_{j.k}a_{k,l}.$$

For $\tau \in A^*$, we write as $\tau = {}^t(\tau_1, \ldots, \tau_n)$. We rewrite ($\beta 1$) as:

$$\begin{aligned}\sum_{(i,j)\in E^1} \tau((\chi_{(i,j)}|\phi(f)\chi_{(i,j)})_A) &= \sum_{(i,j)\in E^1}\sum_{k=1}^{n}\tau_k\sum_{l=1}^{n} f_l\delta_{l,i}\delta_{j,k}a_{k,l} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n}\tau_k f_l a_{k,l} = \sum_{l=1}^{n}\left(\sum_{k=1}^{n} a_{k,l}\tau_k\right) f_l.\end{aligned}$$

We put $B = {}^tD$. Let τ be a tracial state on A. Then ($\beta 1$) and ($\beta 2$) are written as

$$(B\tau|f) = e^{\beta}(\tau|f) \qquad f \in \mathrm{C}(E_r^0) \tag{8}$$

$$(B\tau|f) \le e^{\beta}(\tau|f) \qquad f \in \mathrm{C}(E^0)^+. \tag{9}$$

Lemma 4.2. *If E is a finite graph, ($\beta 1$) implies ($\beta 2$).*

Proof. If E has no sink, the Lemma is trivial.

Let i be a sink and put $f = \chi_i$. The left hand side of (9) is

$$\sum_{l=1}^{n} a_{k,l}f_l = a_{k,i}.$$

If i is a sink then it becomes 0. □

If E has no sink the conditions ($\beta 1$) and ($\beta 2$) are made into one condition:

$$(\beta) \qquad (B\tau|f) = e^{\beta}(\tau|f) \qquad f \in A.$$

Then τ is a Perron-Frobenius eigenvector and e^{β} is the Perron-Frobenius eigenvalue of B.

We assume that E has a sink. We denote by E_1^0 the set of vertices such that there exists an infinite path from them, and denote by E_4^0 the set of vertices such that there exists no infinite path from them. We note that E_1^0 is not empty if and only if the graph E has a loop because E is a finite graph.

We note that $E^0 = E_1^0 \cup E_4^0$ and $E_1^0 \cap E_4^0 = \emptyset$. The set E_4^0 contains all sinks, and all paths which start from the vertices in E_4^0 must end at sinks.

We note that there exists no edge from vertices in E_4^0 to vertices in E_1^0. If such an edge exists, it holds that there exists an infinite path which starts from a vertex in E_4^0.

Lemma 4.3. *We can number E^0 as follows:*

(1) *The numbers of vertices in E_4^0 are larger than that of vertices in E_1^0.*
(2) *There exists no edge from j to i where $i < j$ and i and j are edges in E_4^0.*
(3) *The numbers of sinks are larger than the number of vertices which are not sinks.*

Proof. We note that E_4^0 contains sinks. Fist, we number vertices of E so that the numbers of sinks are larger than the numbers of vertices which are not sink. We denote by $E(1)$ the graph obtained by removing sinks and edges whose ranges are sinks. If $E(1)$ has no sink, the proof is completed. If $E(1)$ has sinks, we renumber vertices of $E(1)$ so that the numbers of sinks in $E(1)$ are larger than the numbers of vertices which are not sinks. We get graphs $E(0)$, $E(1)$, $E(2)$, ..., inductively. We has put $E(0) = E$ for the convenience. Since E is a finite graph, there exists a non negative integer r such that $E(r)$ contains a sink and $E(r+1)$ is empty or $E(r+1)$ has no sink.

The vertices in E_1^0 are not removed, because there exists a infinite path from starting from vertices in E_1^0. On the other hand, vertices in E_4^0 are removed because a path starting from vertex in E_4^0 must reaches a sink, and if such a path remains, then a sink is also remained. $\square$

We denote by F the graph obtained by removing vertices in E_4^0 and edges whose ranges are in E_4^0. We call F the core of the graph E.

We put $E_3^0 = E_s^0$ and put $E_2^0 = E_4^0 \backslash E_3^0$. Then E^0 is expressed as the disjoint union of E_1^0, E_2^0 and E_3^0. Using this dividing of vertices, we write $f = {}^t[f_1\ f_2\ f_3]$, $f_i \in C(E_i^0)$ $(i = 1, 2, 3)$ for $f \in A = C(E^0)$, and $\tau = {}^t[\tau_1\ \tau_2\ \tau_3]$ for a tracial state τ of A. We rewrite $(\beta 1)$ using the above block expression. We use the notation (τ, f) of dual paring instead of $\tau(f)$.

We write the following equation

$$(B\tau, f) = e^{\beta}(\tau, f) \qquad f \in C(E_r^0).$$

using the above block notation as follows:

$$\left(\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ O & B_{22} & B_{23} \\ O & O & O \end{bmatrix}\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}\right) = \left(\begin{bmatrix} e^{\beta}\tau_1 \\ e^{\beta}\tau_2 \\ e^{\beta}\tau_3 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}\right).$$

Since f_1 and f_2 are arbitrary, we have

$$B_{11}\tau_1 + B_{12}\tau_2 + B_{13}\tau_3 = e^{\beta}\tau_1 \tag{10}$$

$$B_{22}\tau_2 + B_{23}\tau_3 = e^{\beta}\tau_2. \tag{11}$$

From (11), we have

$$\tau_2 = e^{-\beta}(B_{22}\tau_2 + B_{23}\tau_3).$$

Since (i,j) element in B_{22} is 0 for $i > j$, we can determine all elements of τ_2 for a given nonnegative τ_3 for every $\beta > 0$.

We assume E has a loop, and E_1^0 is not empty. We note that B_{11} is the transpose of the adjacency matrix of F. Let λ_0 be the Perron-Frobenius eigenvalue of B_{11}. Using (10), we have

$$(e^{\beta} I - B_{11})\tau_1 = B_{12}\tau_2 + B_{13}\tau_3.$$

If e^{β} is greater than λ_0, for nonnegative τ_2, τ_3 we can determine nonnegative τ_1 by

$$\tau_1 = (e^{\beta} I - B_{11})^{-1}(B_{12}\tau_2 + B_{13}\tau_3).$$

For a sink v, let τ_3 be the state corresponding to the Dirac measure δ_v on v. we can determine τ_2 and τ_1, and we can get a tracial state τ_v by the normalization. The tracial state τ_v on A gives the β-KMS state φ_v on $\mathcal{O}_E$.

We summarize the results as the following theorem.

Theorem 4.4. *(1) We assume that E has a loop. We denote by λ_0 the Perron-Frobenius eigenvalue of the transposed of the adjacency matrix of the core F. If $\beta > \log \lambda_0$, the set of the extreme β-KMS states on $\mathcal{O}_E$ with respect to the gauge action correspond to the Dirac measures on the set of sinks in E.*
(2) We assume that E has no loop. Then for every $\beta > 0$, the set of the extreme β-KMS states on $\mathcal{O}_E$ with respect to the gauge action correspond to the Dirac measures on the set of sinks in E.

Proposition 4.5. *Each extreme KMS state in Theorem 4.4 generates a type I factor.*

Proof. We write as $\tau = {}^t\begin{bmatrix}\tau_1 & \tau_2 & \tau_3\end{bmatrix}$. Using the equation (10) and (11), we can write as

$$(I - e^{-\beta}B)\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ \tau_3.\end{bmatrix}$$

Then we have

$$\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix} = (I - e^{-\beta}B)^{-1}\begin{bmatrix}0\\ 0\\ \tau_3.\end{bmatrix}$$

If $\beta > \log \lambda_0$, then $\sum_{i=0}^{\infty} e^{i\beta}B^i$ is convergent, and we have

$$\tau = \sum_{i=0}^{\infty} e^{i\beta}B^i \begin{bmatrix}0\\ 0\\ \tau_3.\end{bmatrix}.$$

Let v be a sink and τ_3 be the state corresponding to the Dirac measure δ_v on v. We can determine τ_2 and τ_1, and we get a tracial state τ_v by the normalization of τ.

The β-KMS state φ_v extending τ_v is of finite type in [23], and generates type I factor ([10]). □

We assume that E has a loop and E_1^0 is not empty. If $\beta = \log \lambda_0$, there exists a β-KMS state on $\mathcal{O}_E$ which is of infinite type in [23]. These KMS state are essentially the same as that given in [4].

Proposition 4.6. *We assume $\beta = \log \lambda_0$. Let $\hat{\tau}_1$ be the normalized Perron Frobenius eigenvector of B_{11}. Then $(\hat{\tau}_1, 0, 0)$ is a β-KMS state on $\mathcal{O}_E$. It corresponds to a β-KMS state of the graph C^*-algebra associated graph F.*

The KMS states in Proposition 4.6 generates type III von Neumann algebra under some condition ([4]).

Remark 4.1. *KMS states on Exel-Laca algebras are classified in* [7] *and graph C^*-algebras are known to be strongly Morita equivalent to some Exel-Laca algebra by adding tails to sinks. But KMS states of finite graphs with sinks can not be obtained directly from that of Exel-Laca algebras because the relation of KMS states on strongly Morita equivalent C^*-algebras are not known.*

References

[1] B. Blackadar, *Operator Algebras,* Springer, 2006.

[2] J. Cuntz, *Simple C^*-algebras generated by isometires,* Commun. Math. Phys. **57** (1977), 173-185.

[3] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains,* Invent. Math. **56** (1980), 251-268.

[4] M. Enomoto, M. Fujii and Y. Watatani, *KMS states for gauge action on O_A,* Math. Japon. **29** (1984), 607–619.

[5] D. Evans, *On O_n,* Publ. Res. Inst. Math. Sci. Kyoto Univ. **16** (1980), 915-927.

[6] R. Exel, *Crossed-products by finite index endomorphisms and KMS states,* J. Funct. Anal. **199** (2003), 153–183.

[7] L. Exel and M. Laca, *Partial dynamical systems and the KMS condition,* Comm. Math. Phys. **232** (2003), 223–277.

[8] N. J. Fowler, M. Laca and I. Raeburn *The C^*-algebras for infinite graphs*, Proc. Amer. Math. Soc. **128** (200), 2319–2327.

[9] N. J. Fowler, P. S. Muhly and I. Raeburn, *Representations of Cuntz-Pimsner Algebras*, Indiana Univ. Math. J. **52** (2003), 569–605.

[10] M. Izumi, T. Kajiwara and Y. Watatani, *KMS states and branched points,* Ergodic Theory Dynam. Systems **27** (2007), 1887–1918.

[11] T. Kajiwara, *Countable bases for Hilbert C^*-modules and classification of KMS states,* Operator Structures and Dynamical Systems, Contemporary Mathematics, **503** (2009),73–91.

[12] T. Kajiwara, C. Pinzari and Y. Watatani, *Ideal structure and simplicity of the C^*–algebras generated by Hilbert bimodules,* J. Funct. Anal. **159** (1998), 295-322.

[13] T. Kajiwara, C. Pinzari and Y. Watatani, *Jones index theory for Hilbert C^*-bimodules and its equivalence with conjugation theory,* J. Funct. Anal., **215** (2004), 1-49.

[14] A. Kumjian and J. Renalut, *KMS states on C^*-algebras associated to expansive maps,* Proc. Amer. Math. Soc., **134** (2006), 2067-2078.

[15] T. Kajiwara and Y. Watatani, *C^*-algebras associated with complex dynamical systems,* Indiana Math. J., **54** (2005), 755–778.

[16] T. Kajiwara and Y. Watatani, *C^*-algebras associated with complex dynamical systems II,* in preparation

[17] T. Kajiwara and Y. Watatani, *KMS states on C^*-algebras associated with self-similar sets* [arXiv : math.OA/0405514]
[18] T. Katsura, *A construction of C^*-algebras from C^*-correspondences,* Advances in quantum dynamics, Contemp. Math. **335** (2005), 173–182.
[19] T. Katsura, *On C^*-algebras associated with C^*-correspondences,* J. Funct. Anal. **217** (2004), 366–401.
[20] T. Katsura *Ideal structure of C^*-algebras associated with C^*-correspondences.* Pacific. J. Math. **230** (2007), 107–145.
[21] A. Kumjian, D. Pask and I. Raeburn, *Cuntz-Krieger algebras of directed graphs,* Pacific J. Math. **184** (1998),161–174.
[22] A.Kumjian, D.Pask, I. Raeburn and J. Renault, *Graphs, groupoids and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541.
[23] M. Laca and S. Neshveyev, *KMS states of quasi-free dynamics on Pimsner algebras,* J. Funct. Anal. **211** (2004) 457–482
[24] K. Matsumoto, Y. Watatani and M. Yoshida, *KMS states for gauge actions on C^*-algebras associated with subshifts,* Math. Z., **228** (1998), 489-509
[25] P.S. Muhly and B. Solel, *Tensor algebras over C^*-correspondences: Representations, Dilations, and C^*-envelopes,* J. Funct. Anal.,**158** (1998), 389-457
[26] D. Olsen and G. K. Pedersen, *Some C^*-dynamical systems with a single KMS state,* Math. Scand. **42** (1978), 111-118
[27] M. Pimsner, *A class of C^*-algebras generating both Cuntz-Krieger algebras and crossed product by $\mathbb{Z}$,* Free probability theory, AMS, (1997), 189–212.
[28] C. Pinzari, Y. Watatani and K. Yonetani, *KMS States, Entropy and the variational principle in full C^*-dynamical systems,* Comm. Math. Phys. **213** (2000), 331–379
[29] I. Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics **103**, Amer. Math. Soc. 2005.

(Tsuyoshi Kajiwara) Department of Environmental and Mathematical Sciences, Okayama University, Tsushima, 700-8530, Japan

(Yasuo Watatani) Department of Mathematical Sciences, Kyushu University, Motooka, Fukuoka, 819-0395, Japan